Exercise 1.3

Question : 02

Find the multiplicative inverse of the following numbers

$\cdot - i - \cdot$

$-3i$

solution

let,

$$z = -3i$$

$$\frac{1}{z} = \frac{1}{-3i} \times \frac{3i}{3i}$$

$$= \frac{3i}{-9i^2}$$

$\because i^2 = -1$

$$= \frac{3i}{-9(-1)}$$

$$= \frac{3i}{9}$$

$= \frac{1}{3} i$

(iii)

$1-2i$

Solution

let,

$$z = 1-2i$$

$$\frac{1}{z} = \frac{1}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{1+2i}{(1-2i)(1+2i)}$$

$\because (a-b)(a+b) = a^2-b^2$

$$= \frac{1+2i}{1-(2i)^2}$$

$$= \frac{1+2i}{1-4i^2}$$

$\because i^2 = -1$

$$= \frac{1+2i}{1-4(-1)}$$

$$= \frac{1+2i}{1+4}$$

$$= \frac{1+2i}{5}$$

$$= \frac{1}{5} + \frac{2}{5} i$$

(iii)

$-3-5i$

solution

let

$$z = -3-5i$$

$$\frac{1}{z} = \frac{1}{-3-5i} \times \frac{-3+5i}{-3+5i}$$

$$= \frac{-3+5i}{(-3-5i)(-3+5i)}$$

$\because (a-b)(a+b) = a^2-b^2$

$$= \frac{-3+5i}{(-3)^2-(5i)^2}$$

$$= \frac{-3+5i}{9-25i^2} \qquad \because i^2=-1$$

$$= \frac{-3+5i}{9-25(-1)}$$

$$= \frac{-3+5i}{9+25}$$

$$= \frac{-3+5i}{34}$$

$$= \frac{-3}{34} + \frac{5}{34}i$$

(iv)

(1, 2)

Solution

Let

$$z = 1+2i$$

$$\frac{1}{z} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{1-2i}{(1+2i)(1-2i)}$$

$$= \frac{1-2i}{1-(2i)^2}$$

$$= \frac{1-2i}{1-4i^2}$$

$\because i^2 = -1$

$$= \frac{1-2i}{1-4(-1)}$$

$$= \frac{1-2i}{1+4}$$

$$= \frac{1-2i}{5}$$

$$= \frac{1}{5} - \frac{2}{5}i$$

$$= \left(\frac{1}{5}, \frac{-2}{5}\right)$$

Question : 03

Simplify

(i)

i^{101}

Solution

$= i^{101}$

$= i^{100} \cdot i^{1}$

$= i \cdot (i^{2})^{50}$ $\because i^{2} = -1$

$= i \cdot (-1)^{50}$

$= i \cdot (1)$

$= i$

(ii)

$(-ai)^{4}$

Solution

$= (-ai)^{4}$

$= (-1)^{4} \cdot (a)^{4} \cdot i^{4}$

$= 1 \cdot a^{4} \cdot (i^{2})^{2}$ $\because i^{2} = -1$

$= a^{4} \cdot (-1)^{2}$

$= a^{4}$

(iii)

i^{-3}

Solution

$= i^{-3}$

$= \frac{1}{i^3}$

$= \frac{1}{i \cdot i^2}$

$= \frac{1}{i \cdot (-1)}$ $\quad \because i^2 = -1$

$= \frac{1}{-i} \times \frac{i}{i}$

$= \frac{i}{-i^2}$

$= \frac{i}{-(-1)}$

$= \frac{i}{1}$

$= i$

-(iv)-

i^{-10}

Solution

$= i^{-10}$

$= \frac{1}{i^{10}}$

$= \frac{1}{(i^2)^5}$ $\because i^2 = -1$

$= \frac{1}{(-1)^5}$

$= -1$

Question : 04

prove that

$\bar{z} = z$ iff z is a real number

Solution

let

$z = a+bi$ Then

$\bar{z} = a-bi$

Suppose,

$z = \bar{z}$

$a + bi = a - bi$

$a + bi - a + bi = 0$

$2bi = 0 \qquad \because 2i \neq 0$

$b = 0$

$\Rightarrow z = a + 0i$

$z = a$

Hence, z is real.

Now Conversaly,

Suppose,

z is real such That

$z = a$

Then $\bar{z} = a$

$\Rightarrow z = \bar{z}$

Hence proved

Question: 05

Simplify by expressing in the form of $a + bi$

(i)

$5+2\sqrt{-4}$

Solution

$= 5+2\sqrt{-4}$

$= 5+2\sqrt{-4}$

$= 5+2\sqrt{-1}\cdot\sqrt{4}$ $\quad \because i=\sqrt{-1}$

$= 5+2(i)(2)$

$= 5+4i$

(ii)

$(2+\sqrt{-3})(3+\sqrt{-3})$

Solution

$= (2+\sqrt{-3})(3+\sqrt{-3})$ $\quad \because i=\sqrt{-1}$

$= (2+\sqrt{-1}\cdot\sqrt{3})(3+\sqrt{-1}\cdot\sqrt{3})$

$= (2+\sqrt{3}\,i)(3+\sqrt{3}\,i)$ $\quad \because i^2=-1$

$= 6+2\sqrt{3}\,i+3\sqrt{3}\,i+(\sqrt{3}\,i)^2$

$= 6 + 5\sqrt{3}i + 3i^2$

$= 6 + 5\sqrt{3}i + 3(-1)$

$= 6 + 5\sqrt{3}i - 3$

$= 3 + 5\sqrt{3}i$

iii

$$\frac{2}{\sqrt{5} + \sqrt{-8}}$$

Solution

$$= \frac{2}{\sqrt{5} + \sqrt{-8}}$$

$$= \frac{2}{\sqrt{5} + \sqrt{-1}\cdot\sqrt{8}}$$

$$= \frac{2}{\sqrt{5} + \sqrt{8}i} \qquad \because i = \sqrt{-1}$$

$$= \frac{2}{\sqrt{5} + \sqrt{8}i} \times \frac{\sqrt{5} - \sqrt{8}i}{\sqrt{5} - \sqrt{8}i}$$

$$= \frac{2(\sqrt{5} - \sqrt{8}i)}{(\sqrt{5} + \sqrt{8}i)(\sqrt{5} - \sqrt{8}i)}$$

$\because (a+b)(a-b) = a^2 - b^2$

$$= \frac{2\sqrt{5} - 2\sqrt{8}i}{(\sqrt{5})^2 - (\sqrt{8}i)^2}$$

$= \frac{2\sqrt{5} - 2\sqrt{8}i}{5 - 8i^2}$ $\because i^2 = -1$

$= \frac{2\sqrt{5} - 2\sqrt{8}i}{5 - 8(-1)}$

$= \frac{2\sqrt{5} - 2\sqrt{8}i}{5+8}$

$= \frac{2\sqrt{5} - 2\sqrt{8}i}{13}$

$= \frac{2\sqrt{5}}{13} - \frac{2\sqrt{8}}{13}i = \frac{2\sqrt{5}}{13} - \frac{2\sqrt{4\times2}}{13}i$

$= \frac{2\sqrt{5}}{13} - \frac{4\sqrt{2}}{13}i$

(iv)

$\frac{3}{\sqrt{6} - \sqrt{-12}}$

Solution

$= \frac{3}{\sqrt{6} - \sqrt{-12}}$

$= \frac{3}{\sqrt{6} - \sqrt{-1}\cdot\sqrt{12}}$ $\because i = -1$

$= \frac{3}{\sqrt{6} - \sqrt{12}i} \times \frac{\sqrt{6} + \sqrt{12}i}{\sqrt{6} + \sqrt{12}i}$

$\because (a+b)(a-b) = a^2 - b^2$

$$= \frac{3(\sqrt{6}+\sqrt{12}\,i)}{(\sqrt{6})^2-(\sqrt{12}\,i)^2}$$

$$= \frac{3\sqrt{6}+3\sqrt{12}\,i}{6-12i^2} \qquad \because i^2=-1$$

$$= \frac{3\sqrt{6}+3\sqrt{4\times 3}\,i}{6-12(-1)}$$

$$= \frac{3\sqrt{6}+6\sqrt{3}\,i}{6+12}$$

$$= \frac{3\sqrt{6}+6\sqrt{3}\,i}{18}$$

$$= \frac{3(\sqrt{6}+2\sqrt{3}\,i)}{18}$$

$$= \frac{\sqrt{6}+2\sqrt{3}\,i}{6}$$

$$= \frac{\sqrt{6}}{6} + \frac{\cancel{2}\sqrt{3}}{3\cancel{6}}\,i$$

$$= \frac{\sqrt{6}}{\sqrt{6}\times\sqrt{6}} + \frac{\sqrt{3}}{\sqrt{3}\times\sqrt{3}}\,i$$

$$= \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\,i$$

Question : 06

Show that $\forall z \in \mathbb{C}$

~~4ib~~

$z^2 + \bar{z}^2$ is a real number

Solution

Let

$$z = a + bi$$

$$\bar{z} = a - bi$$

$$z^2 = (a+bi)^2$$

$$z^2 = a^2 + b^2 i^2 + 2abi$$

$$z^2 = a^2 + b^2(-1) + 2abi \quad \because i^2 = -1$$

$$z^2 = a^2 - b^2 + 2abi$$

$$\bar{z}^2 = (a - bi)^2$$

$$\bar{z}^2 = a^2 + b^2 i^2 - 2abi$$

$$\bar{z}^2 = a^2 + b^2(-1) - 2abi$$

$$\bar{z}^2 = a^2 - b^2 - 2abi$$

$$z^2 + \bar{z}^2 = a^2 - b^2 + \cancel{2abi} + a^2 - b^2 - \cancel{2abi}$$

$$z^2 + \bar{z}^2 = 2a^2 - 2b^2$$

$$z^2 + \bar{z}^2 = 2(a^2 - b^2)$$

which is real number

Hence proved

(iii)

$(z-\bar{z})^2$ is a real number

Solution

let,

$$z = a+bi$$

$$\bar{z} = a-bi$$

$$(z-\bar{z})^2 = (a+bi-(a-bi))^2$$

$$= (a+bi-a+bi)^2$$

$$(z-\bar{z})^2 = (2bi)^2$$

$$(z-\bar{z})^2 = 4b^2i^2$$

$$(z-\bar{z})^2 = 4b^2(-1)$$

$$(z-\bar{z})^2 = -4b^2$$

which is a real number

Hence proved

Question : 07

Simplify the following

(i)

$$\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)^3$$

Solution

$$=\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)^3$$

$$=\left(\frac{-1+\sqrt{3}i}{2}\right)^3$$

$$=\frac{(-1+\sqrt{3}i)^3}{(2)^3}$$

$$\because (a+b)^3=a^3+b^3+3ab(a+b)$$

$$=\frac{(-1)^3+(\sqrt{3}i)^3+3(-1)(\sqrt{3}i)(-1+\sqrt{3}i)}{8}$$

$$=\frac{-1+(\sqrt{3}i)^2(\sqrt{3}i)-3\sqrt{3}i(-1+\sqrt{3}i)}{8}$$

$$=\frac{-1+3i^2(\sqrt{3}i)+3\sqrt{3}i-3(\sqrt{3}i)^2}{8}$$

$$=\frac{-1+3(-1)(\sqrt{3}i)+3\sqrt{3}i-3(3i^2)}{8}$$

$= \frac{-1 - 3\sqrt{3}i + 3\sqrt{3}i - 9(-1)}{8}$

$= \frac{-1+9}{8}$

$= \frac{8}{8}$

$= 1$

(iii)

$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

Solution

$= \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

$= \left(\frac{-1-\sqrt{3}i}{2}\right)^3$

$= \frac{(-1-\sqrt{3}i)^3}{(2)^3}$

$= \frac{(-1-\sqrt{3}i)^3}{8}$

$\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$$= \frac{(-1)^3 - (\sqrt{3}i)^3 - 3(-1)(\sqrt{3}i)(-1-\sqrt{3}i)}{8}$$

$$= \frac{-1 - (\sqrt{3}i)^2(\sqrt{3}i) + 3\sqrt{3}i(-1-\sqrt{3}i)}{8}$$

$$= \frac{-1 - 3(-1)(\sqrt{3}i) + 3\sqrt{3}i(-1-\sqrt{3}i)}{8}$$

$$= \frac{-1 + 3\sqrt{3}i - 3\sqrt{3}i - 3(\sqrt{3}i)^2}{8}$$

$$= \frac{-1-3\cdot 3i^2}{8} \qquad \because i^2 = -1$$

$$= \frac{-1-9(-1)}{8}$$

$$= \frac{-1+9}{8} = \frac{8}{8}$$

$$= 1$$

(iii)

$$\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$$

Solution

$$=\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$=\left(\frac{-1-\sqrt{3}i}{2}\right)^{-2}\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$=\left(\frac{2}{-1-\sqrt{3}i}\right)^{2}\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$=\left(\frac{(2)^2}{(-1-\sqrt{3}i)^2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$\because (a-b)^2 = a^2+b^2-2ab$$

$$=\left(\frac{4}{1+(\sqrt{3}i)^2-2(-1)(\sqrt{3}i)}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$=\left(\frac{4}{1+3(-1)+2\sqrt{3}i}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$=\left(\frac{4}{1-3+2\sqrt{3}i}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$=\left(\frac{4}{-2+2\sqrt{3}i}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= \left(\frac{4}{2(-1+\sqrt{3}i)}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= \frac{\cancel{2}}{-1+\sqrt{3}i} \times \frac{-1-\sqrt{3}i}{\cancel{2}}$$

$$= \frac{-1-\sqrt{3}i}{-1+\sqrt{3}i} \times \frac{-1-\sqrt{3}i}{-1-\sqrt{3}i}$$

$$= \frac{(-1-\sqrt{3}i)^2}{(-1)^2-(\sqrt{3}i)^2}$$

$$= \frac{1+(\sqrt{3}i)^2-2(-1)(\sqrt{3}i)}{1-3i^2}$$

$$= \frac{1+3i^2+2\sqrt{3}i}{1-3(-1)} \quad \because i^2=-1$$

$$= \frac{1+3(-1)+2\sqrt{3}i}{1+3}$$

$$= \frac{1-3+2\sqrt{3}i}{4}$$

$$= \frac{-2+2\sqrt{3}i}{4}$$

$$= \frac{\cancel{2}(-1+\sqrt{3}i)}{\cancel{4}\,2}$$

$$= \frac{-1+\sqrt{3}i}{2}$$

(iv)

$(a+bi)^2$

Solution

$= (a+bi)^2$

$\because (a+b)^2 = a^2+b^2+2ab$

$= a^2+b^2i^2+2abi \quad \because i^2=-1$

$= a^2+b^2(-1)+2abi$

$= a^2-b^2+2abi$

$= a^2+2abi-b^2$

(v)

$(a+bi)^{-2}$

Solution

$= (a+bi)^{-2}$

$= \dfrac{1}{(a+bi)^2}$

$\because (a+b)^2 = a^2+2ab+b^2$

$= \dfrac{1}{a^2+b^2i^2-2abi} \quad \because i^2=-1$

$= \dfrac{1}{a^2+b^2(-1)-2abi}$

$$= \frac{1}{a^2 + 2abi - b^2}$$

$$= \frac{1}{a^2 - b^2 + 2abi} \times \frac{a^2 - b^2 - 2abi}{a^2 - b^2 - 2abi}$$

$$= \frac{a^2 - b^2 - 2abi}{(a^2 - b^2)^2 - (2abi)^2}$$

$$= \frac{a^2 - b^2 - 2abi}{(a^2 - b^2)^2 - (2abi)^2}$$

$$= \frac{a^2 - b^2 - 2abi}{(a^2 - b^2)^2 - 4a^2b^2i^2}$$

$$= \frac{a^2 - b^2 - 2abi}{(a^2 - b^2)^2 - 4a^2b^2(-1)}$$

$$= \frac{a^2 - b^2 - 2abi}{a^4 + b^4 - 2a^2b^2 + 4a^2b^2}$$

$$= \frac{a^2 - b^2 - 2abi}{a^4 + b^4 + 2a^2b^2}$$

$$= \frac{a^2 - b^2 - 2abi}{(a^2 + b^2)^2}$$

$$= \frac{a^2 - b^2}{(a^2 + b^2)^2} - \frac{2ab}{(a^2 + b^2)^2} i$$

(vi)

$(a+bi)^3$

Solution

$= (a+bi)^3$

$\because (a+b)^3 = a^3+b^3+3ab(a+b)$

$= a^3 + b^3i^3 + 3abi(a+bi)$

$= a^3 + b^3i \cdot i^2 + 3a^2bi + 3ab^2i^2 \quad \because i^2=-1$

$= a^3+b^3i(-1) + 3a^2bi + 3ab^2(-1)$

$= a^3 - b^3i + 3a^2bi - 3ab^2$

$= a^3 - 3ab^2 + (3a^2b - b^3)i$

(vii)

$(a-bi)^3$

Solution

$= (a-bi)^3$

$\because (a-b)^3 = a^3-b^3-3ab(a-b)$

$= a^3 - b^3i^3 - 3abi(a-bi)$

$= a^3 - b^3i \cdot i^2 - 3a^2bi + 3ab^2i^2 \quad \because i^2=-1$

$= a^3 - b^3i(-1) - 3a^2bi + 3ab^2(-1)$

$= a^3 + b^3i - 3a^2bi - 3ab^2$

$= a^3 - 3ab^2 + (b^3 - 3a^2b)i$

Q(viii)

$(3-\sqrt{-4})^{-3}$

Solution

$$= (3-\sqrt{-4})^{-3}$$

$$= \frac{1}{(3-\sqrt{-4})^3}$$

$$= \frac{1}{(3-\sqrt{4}i)^3}$$

$$= \frac{1}{(3-\sqrt{4}\,i)^3}$$

$\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$$= \frac{1}{27-(\sqrt{4}i)^3 - 3(3)(\sqrt{4}i)(3-\sqrt{4}i)}$$

$$= \frac{1}{27-(\sqrt{4}i)^2\cdot\sqrt{4}i - 9\sqrt{4}i(3-\sqrt{4}i)}$$

$\because i^2=-1$

$$= \frac{1}{27-4(-1)\cdot\sqrt{4}i - 27\sqrt{4}i + 9(\sqrt{4}i)^2}$$

$$= \frac{1}{27+4\sqrt{4}i - 27\sqrt{4}i + 36i^2}$$

$$= \frac{1}{27-23\sqrt{4}i+36(-1)}$$

$$= \frac{1}{27-36-23\sqrt{4}i}$$

$$= \frac{1}{-9-23\sqrt{4}i} \times \frac{-9+23\sqrt{4}i}{-9+23\sqrt{4}i}$$

$$= \frac{-9+23\sqrt{4}i}{(-9)^2-(23\sqrt{4}i)^2}$$

$$= \frac{-9+23\sqrt{4}i}{81-529\times4i^2}$$

$$= \frac{-9+23\sqrt{4}i}{81-2116(-1)}$$

$$= \frac{-9+23\sqrt{4}i}{81+2116}$$

$$= \frac{-9+23\sqrt{4}i}{2197}$$

$$= \frac{-9+23(2)i}{2197}$$

$$= \frac{-9+46i}{2197}$$

$$= \frac{-9}{2197} + \frac{46}{2197}i$$