Ex # 2.4

① Find $\frac{dy}{dx}$ using substitution method.

i) $y = \sqrt{\frac{1-x}{1+x}}$

let

$$\frac{1-x}{1+x} = t$$

Diff. w.r.t 'x'

$$\frac{d}{dx}\left(\frac{1-x}{1+x}\right) = \frac{dt}{dx}$$

$$\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2} = \frac{dt}{dx}$$

$$\frac{(1+x)(0-1)-(1-x)(0+1)}{(1+x)^2} = \frac{dt}{dx}$$

$$\frac{-1-\cancel{x}-1+\cancel{x}}{(1+x)^2} = \frac{dt}{dx}$$

$$\frac{-2}{(1+x)^2} = \frac{dt}{dx}$$

Then

$$y = \sqrt{t}$$

Diff w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt} t^{\frac{1}{2}}$$

$$= \frac{1}{2} t^{\frac{1}{2}-1} \frac{d}{dt}(t)$$

$$= \frac{1}{2} t^{-\frac{1}{2}} (1)$$

$$= \frac{1}{2\sqrt{t}}$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{1}{2\sqrt{t}} \times \frac{-2}{(1+x)^2}$$

By putting the value of 't'

$$= \frac{1}{\cancel{2}\sqrt{\frac{1-x}{1+x}}} \cdot \frac{-\cancel{2}}{(1+x)^2}$$

$$= \frac{-1}{\sqrt{1-x}\,(1+x)^{2-\frac{1}{2}}}$$

$$= \frac{-1}{\sqrt{1-x}\,(1+x)^{3/2}}$$

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iii) $y = \sqrt{x+\sqrt{x}}$

let

$$t = x+\sqrt{x}$$

Diff w.r.t 'x'

$$\frac{dt}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}x^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^{\frac{1}{2}-1}\frac{d}{dx}(x)$$

$$= 1 + \frac{1}{2}x^{-\frac{1}{2}}(1)$$

$$= 1 + \frac{1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

Then,

$$y = \sqrt{t}$$

Diff — w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt} t^{\frac{1}{2}}$$

$$\frac{dy}{dt} = \frac{1}{2} t^{\frac{1}{2}-1} \frac{d}{dt}(t)$$

$$= \frac{1}{2} t^{\frac{-1}{2}}(1)$$

$$= \frac{1}{2\sqrt{t}}$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{1}{2\sqrt{t}} \times \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

By putting the value of 't'

$$= \frac{1}{2\sqrt{x+\sqrt{x}}} \times \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$$

iii) $y = x\sqrt{\frac{a+x}{a-x}}$

Let

$$t = \frac{a+x}{a-x}$$

Diff w.r.t. 'x'

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2}$$

$$= \frac{a-\cancel{x}+a+\cancel{x}}{(a-x)^2}$$

$$\frac{dt}{dx} = \frac{2a}{(a-x)^2}$$

Then

$$y = x\sqrt{t}$$

Diff w.r.t 't'

$$\frac{dy}{dt} = \frac{d}{dt}\, x\, t^{\frac{1}{2}}$$

$$= x\frac{d}{dt}(t^{\frac{1}{2}}) + t^{\frac{1}{2}}\frac{d}{dt}(x)$$

$$\frac{dy}{dt} = x\,\frac{1}{2}t^{\frac{1}{2}-1}\frac{d}{dt}(t) + t^{\frac{1}{2}}\,\frac{(a-x)^2}{2a}$$

$$\frac{dy}{dt} = \frac{x}{2\sqrt{t}} + \sqrt{t}\,\frac{(a-x)^2}{2a}$$

$$= \frac{xa + (\sqrt{t})^2(a-x)^2}{2a\sqrt{t}}$$

$$= \frac{xa + t(a-x)^2}{2a\sqrt{t}}$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{xa + t(a-x)^2}{2a\sqrt{t}} \times \frac{2a}{(a-x)^2}$$

By putting the value of 't'

$$= \frac{xa + \left(\frac{a+x}{a-x}\right)(a-x)^2}{\cancel{2a}\sqrt{\frac{a+x}{a-x}}} \times \frac{\cancel{2a}}{(a-x)^2}$$

$$= \frac{xa + (a+x)(a-x)}{\frac{\sqrt{a+x}\,(a-x)^2}{\sqrt{a-x}}}$$

$$= \frac{xa + a^2 - x^2}{\sqrt{a+x}\,(a-x)^{2-\frac{1}{2}}} = \frac{a^2 - x^2 + xa}{\sqrt{a+x}\,(a-x)^{3/2}}$$

iv) $y = (3x^2-2x+7)^6$

Let

$t = 3x^2-2x+7$

Diff w.r.t. 'x'

$\frac{dt}{dx} = 3\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(7)$

$\frac{dt}{dx} = 3\times 2x^{2-1}\frac{d}{dx}(x) - 2(1) + 0$

$= 6x(1) - 2$

$= 6x-2$

Then

$y = t^6$

Diff w.r.t 't'

$\frac{dy}{dt} = \frac{d}{dt}t^6$

$= 6t^{6-1}\frac{d}{dt}(t)$

$= 6t^5(1)$

$= 6t^5$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 6t^5\ (6x-2)$$

By putting the value of 't'

$$= 6\,(3x^2-2x+7)^5\, 2(3x-1)$$

$$= 12\,(3x-1)(3x^2-2x+7)^5$$

v) $y=\sqrt{\frac{a^2+x^2}{a^2-x^2}}$

let

$$t = \frac{a^2+x^2}{a^2-x^2}$$

Differentiate w.r.t 'x'

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{a^2+x^2}{a^2-x^2}\right)$$

$$= \frac{(a^2-x^2)\frac{d}{dx}(a^2+x^2)-(a^2+x^2)\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}$$

$$= \frac{(a^2-x^2)\left[\frac{d}{dx}(a^2)+\frac{d}{dx}(x^2)\right]-(a^2+x^2)\left[\frac{d}{dx}(a^2)-\frac{d}{dx}(x^2)\right]}{(a^2-x^2)^2}$$

$$= \frac{(a^2-x^2)(0+2x)-(a^2+x^2)(0-2x)}{(a^2-x^2)^2}$$

$$= \frac{2x(a^2-\cancel{x^2}+a^2+\cancel{x^2})}{(a^2-x^2)^2}$$

$$= \frac{2x(2a^2)}{(a^2-x^2)^2}$$

$$= \frac{4a^2x}{(a^2-x^2)^2}$$

Then

$$y = \sqrt{t}$$

Differentiate w.r.t. 't'

$$\frac{dy}{dt} = \frac{d}{dt} t^{\frac{1}{2}}$$

$$= \frac{1}{2} t^{\frac{1}{2}-1} \frac{d}{dt}(t)$$

$$= \frac{1}{2} t^{-\frac{1}{2}} (1)$$

$$= \frac{1}{2\sqrt{t}}$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{1}{2\sqrt{t}} \times \frac{4a^2x}{(a^2-x^2)^2}$$

By putting the value of 't'

$$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{\frac{a^2+x^2}{a^2-x^2}}} \times \frac{\cancel{4}^{2}a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{(a^2+x^2)}}{(a^2-x^2)^{\frac{1}{2}}}} \cdot \frac{2a^2x}{(a^2-x^2)^2}$$

$$= \frac{2a^2x}{\sqrt{a^2+x^2}\,(a^2-x^2)^{2-\frac{1}{2}}}$$

$$= \frac{2a^2x}{\sqrt{a^2+x^2}\,(a^2-x^2)^{\frac{3}{2}}}$$

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(2) Find $\frac{dy}{dx}$ ...

i) $3x+4y+7=0$

Differentiate w.r.t 'x'

$$3\frac{d}{dx}(x)+4\frac{d}{dx}(y)+\frac{d}{dx}(7)=0$$

$$3(1)+4\frac{dy}{dx}+0=0$$

$$3+4\frac{dy}{dx}=0$$

$$4\frac{dy}{dx}=-3$$

$$\frac{dy}{dx}=\frac{-3}{4}$$

ii) $xy + y^2 = 2$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2)$$

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + 2y^{2-1}\frac{d}{dx}(y) = 0$$

$$x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$$

$$\frac{dy}{dx}(x+2y) = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x+2y}}$$

iii) $x^2 - 4xy - 5y = 0$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = \frac{d}{dx}(0)$$

$$2x^{2-1}\frac{d}{dx}(x) - 4\left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right] - 5\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx}\,2(1)-4\left[\frac{x\,dy}{dx}+y(1)\right]-5\frac{dy}{dx}=0$$

$$2-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$2-4y=4x\frac{dy}{dx}+5\frac{dy}{dx}$$

$$2-4y=\frac{dy}{dx}(4x+5)$$

$$\boxed{\frac{2-4y}{4x+5}=\frac{dy}{dx}}$$

iv) $4x^2+2hxy+by^2+2gx+2fy+c=0$

Differentiate w.r.t. 'x'

$$4\frac{d}{dx}(x^2)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y^2)+2g\frac{d}{dx}(x)+2f\frac{d}{dx}(y)$$

$$+\frac{d}{dx}(c)=0$$

$$4\times 2x^{2-1}\frac{d}{dx}(x)+2h\left[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\right]+b2y^{2-1}\frac{d}{dx}(y)+2g(1)$$

$$+2f\frac{dy}{dx}=0$$

$$8x+2h\left[x\frac{dy}{dx}+y(1)\right]+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$8x+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$2hx\frac{dy}{dx}+2by\frac{dy}{dx}+2f\frac{dy}{dx}=-8x-2hy-2g$$

$$\frac{dy}{dx}2(hx+by+f)=-2(4x+hy+g)$$

$$\frac{dy}{dx}=\frac{-\cancel{2}(4x+hy+g)}{\cancel{2}(hx+by+f)}$$

$$\boxed{\frac{dy}{dx}=-\frac{4x+hy+g}{hx+by+f}}$$

v) $x\sqrt{1+y}+y\sqrt{1+x}=0$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}x(1+y)^{\frac{1}{2}}+\frac{d}{dx}y(1+x)^{\frac{1}{2}}=0$$

$$x\frac{d}{dx}(1+y)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}\frac{d}{dx}(x)+y\frac{d}{dx}(1+x)^{\frac{1}{2}}+(1+x)^{\frac{1}{2}}\frac{d}{dx}(y)=0$$

$$x\frac{1}{2}(1+y)^{\frac{1}{2}-1}\frac{d}{dx}(1+y)+(1+y)^{\frac{1}{2}}(1)+y\cdot\frac{1}{2}(1+x)^{\frac{1}{2}-1}\frac{d}{dx}(1+x)+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\frac{x}{2}(1+y)^{-\frac{1}{2}}\left(0+\frac{dy}{dx}\right)+(1+y)^{\frac{1}{2}}+\frac{y}{2}(1+x)^{-\frac{1}{2}}(0+1)+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\frac{x}{2(1+y)^{\frac{1}{2}}}\frac{dy}{dx}+(1+y)^{\frac{1}{2}}+\frac{y}{2(1+x)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\frac{x}{2(1+y)^{\frac{1}{2}}}\frac{dy}{dx}+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=-(1+y)^{\frac{1}{2}}-\frac{y}{2(1+x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}\left(\frac{x}{2(1+y)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\right)=-\frac{2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}+y}{2(1+x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}\left(\frac{x+2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}{2(1+y)^{\frac{1}{2}}}\right)=-\frac{2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}+y}{2(1+x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=-\frac{2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}+y}{\cancel{2}(1+x)^{\frac{1}{2}}}\times\frac{\cancel{2}(1+y)^{\frac{1}{2}}}{x+2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=-\frac{\left[2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}-y\right](1+y)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}\left[x+2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}\right]}$$

vi) $y(x^2-1) = x\sqrt{x^2+4}$

Differentiate w.r.t. 'x'

$$\frac{d}{dx}y(x^2-1) = \frac{d}{dx}x(x^2+4)^{\frac{1}{2}}$$

$$y\frac{d}{dx}(x^2-1) + (x^2-1)\frac{d}{dx}y = x\frac{d}{dx}(x^2+4)^{\frac{1}{2}} + (x^2+4)^{\frac{1}{2}}\frac{d}{dx}(x)$$

$$y(2x-0) + (x^2-1)\frac{dy}{dx} = x \cdot \frac{1}{\cancel{2}}(x^2+4)^{-\frac{1}{2}}(\cancel{2}x) + (x^2+4)^{\frac{1}{2}}(1)$$

$$y(2x) + (x^2-1)\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2+4}} + \sqrt{x^2+4}$$

$$(x^2-1)\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2+4}} + \sqrt{x^2+4} - 2xy$$

$$\frac{dy}{dx} = \frac{x^2 + (\sqrt{x^2+4})^2 - 2xy\sqrt{x^2+4}}{\sqrt{x^2+4}\,(x^2-1)}$$

$$\frac{dy}{dx} = \frac{x^2 + x^2 + 4 - 2xy\sqrt{x^2+4}}{\sqrt{x^2+4}\,(x^2-1)}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2xy\sqrt{x^2+4}}{\sqrt{x^2+4}\,(x^2-1)}$$

(3) $\frac{dy}{dx} = ?$

i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$

Differentiate w.r.t. 'θ'

$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right)$, $\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1)$

$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1})$, $\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1)$

$\frac{dx}{d\theta} = 1 + (-1)\theta^{-1-1}\frac{d}{d\theta}(\theta)$, $\frac{dy}{d\theta} = 1 + 0$

$\frac{dx}{d\theta} = 1 - \theta^{-2}(1)$, $\frac{dy}{d\theta} = 1$

$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2}$,

$\frac{dx}{d\theta} = \frac{\theta^2 - 1}{\theta^2}$,

Using Chain Rule

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 1 \times \frac{\theta^2}{\theta^2 - 1} = \frac{\theta^2}{\theta^2 - 1}$

(ii) $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{2bt}{1+t^2}$

Differentiate w.r.t. 't'

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right), \quad \frac{dy}{dt} = 2b\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right], \quad \frac{dy}{dt} = 2b\left[\frac{(1+t^2)\frac{d}{dt}(t)-(t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}\right], \quad \frac{dy}{dt} = 2b\left[\frac{(1+t^2)(1)-t(1+2t)}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{-2t(1+\cancel{t^2}+1-\cancel{t^2})}{(1+t^2)^2}\right], \quad \frac{dy}{dt} = 2b\left[\frac{1+t^2-2t^2}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{-2t(2)}{(1+t^2)^2}\right], \quad \frac{dy}{dt} = 2b\left[\frac{1-t^2}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}, \quad \frac{dy}{dt} = 2b\left[\frac{1-t^2}{(1+t^2)^2}\right]$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cancel{2}b\left[\frac{1-t^2}{\cancel{(1+t^2)^2}}\right] \times \left[\frac{\cancel{(1+t^2)^2}}{-\cancel{4}at}\right]$$

$$= -\frac{b(1-t^2)}{2at}$$

(4)

$x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Differentiate w.r.t. 't'

$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$, $\frac{dy}{dt} = 2\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$

$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$, $\frac{dy}{dt} = 2\left[\frac{(1+t^2)\frac{d}{dt}(t) - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$

$\frac{dx}{dt} = \frac{(1+t^2)(0-2t) - (1-t^2)(0+2t)}{(1+t^2)^2}$, $\frac{dy}{dt} = 2\left[\frac{(1+t^2)(1) - t(0+2t)}{(1+t^2)^2}\right]$

$\frac{dx}{dt} = \frac{-2t(1+\cancel{t^2}+1-\cancel{t^2})}{(1+t^2)^2}$, $\frac{dy}{dt} = 2\left[\frac{1+t^2-2t^2}{(1+t^2)^2}\right]$

$\frac{dx}{dt} = \frac{-2t(2)}{(1+t^2)^2}$, $\frac{dy}{dt} = 2\left[\frac{1-t^2}{(1+t^2)^2}\right]$

$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$

Using Chain Rule

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\cancel{2}(1-t^2)}{\cancel{(1+t^2)^2}} \times \frac{\cancel{(1+t^2)^2}}{-\cancel{4}t} = -\frac{(1-t^2)}{2t}$

$$y\frac{dy}{dx}+x=0$$

$$\frac{2t}{1+t^2}x-\frac{(1-t^2)}{2t}+\frac{1-t^2}{1+t^2}=0$$

$$-\frac{1-t^2}{1+t^2}+\frac{1-t^2}{1+t^2}=0$$

$$0=0$$

Hence Proved

⑤ Differentiate

i) $x^2-\frac{1}{x^2}$ w.r.t. x^4

let

$$y=x^2-\frac{1}{x^2}\quad,\quad t=x^4$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{d}{dx}(x^2)-\frac{d}{dx}(x^{-2})\quad,\quad\frac{dt}{dx}=\frac{d}{dx}(x^4)$$

$$\frac{dy}{dx}=2x^{2-1}\frac{d}{dx}(x)-(-2)x^{-2-1}\frac{d}{dx}(x)\quad,\quad\frac{dt}{dx}=4x^{4-1}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = 2x(1) + 2x^{-3}(1) \quad , \quad \frac{dt}{dx} = 4x^3(1)$$

$$= 2x + \frac{2}{x^3} \quad , \quad = 4x^3$$

$$= \frac{2x^4 + 2}{x^3} = \frac{2(x^4+1)}{x^3}$$

Using Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{\cancel{2}(x^4+1)}{x^3} \times \frac{1}{\cancel{4}_{2}x^3}$$

$$= \frac{x^4+1}{2x^6}$$

ii) $(1+x^2)^n$ w.r.t. x^2

Let

$$y = (1+x^2)^n \quad , \quad t = x^2$$

Differentiate w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(1+x^2)^n \quad , \quad \frac{dt}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = n(1+x^2)^{n-1}\frac{d}{dx}(1+x^2) \quad , \quad \frac{dt}{dx} = 2x^{2-1}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = n(1+x^2)^{n-1}(0+2x) \quad , \quad \frac{dt}{dx} = 2x(1)$$

$$\frac{dy}{dx} = 2nx(1+x^2)^{n-1} \quad , \quad \frac{dt}{dx} = 2x$$

Using Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \cancel{2}n\cancel{x}(1+x^2)^{n-1} \times \frac{1}{\cancel{2x}}$$

$$= n(1+x^2)^{n-1}$$

iii) $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

Let

$$y = \frac{x^2+1}{x^2-1} \quad , \quad t = \frac{x-1}{x+1}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dt} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) \quad , \quad \frac{dt}{dx} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$\frac{dy}{dt} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \quad , \quad \frac{dt}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2x+0) - (x^2+1)(2x-0)}{(x^2-1)^2}, \quad \frac{dt}{dx} = \frac{(x+1)(1-0) - (x-1)(1+0)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2x(\cancel{x^2}-1-\cancel{x^2}-1)}{(x^2-1)^2}, \quad \frac{dt}{dx} = \frac{\cancel{x}+1-\cancel{x}+1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2-1)^2}, \quad \frac{dt}{dx} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Using Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{-\cancel{4}^{2}x}{(x^2-1)^2} \times \frac{(x+1)^2}{\cancel{2}_{1}}$$

$$= \frac{-2x\cancel{(x+1)^2}}{\cancel{(x+1)^2}(x-1)^2}$$

$$= \frac{-2x}{(x-1)^2}$$

iv) $\dfrac{ax+b}{cx+d}$ w.r.t. $\dfrac{ax^2+b}{ax^2+d}$

Let

$y = \dfrac{ax+b}{cx+d}$, $t = \dfrac{ax^2+b}{ax^2+d}$

Differentiate w.r.t. 'x'

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{ax+b}{cx+d}\right)$, $\dfrac{dt}{dx} = \dfrac{d}{dx}\left(\dfrac{ax^2+b}{ax^2+d}\right)$

$\dfrac{dy}{dx} = \dfrac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$, $\dfrac{dt}{dx} = \dfrac{(ax^2+d)\frac{d}{dx}(ax^2+b) - (ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$

$\dfrac{dy}{dx} = \dfrac{(cx+d)(a(1)+0) - (ax+b)(c(1)+0)}{(cx+d)^2}$, $\dfrac{dt}{dx} = \dfrac{(ax^2+d)(a(2x)+0) - (ax^2+b)(a(2x)+0)}{(ax^2+d)^2}$

$\dfrac{dy}{dx} = \dfrac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$, $\dfrac{dt}{dx} = \dfrac{(ax^2+d)(2ax) - (ax^2+b)(2ax)}{(ax^2+d)^2}$

$\dfrac{dy}{dx} = \dfrac{acx+d-acx-b}{(cx+d)^2}$, $\dfrac{dt}{dx} = \dfrac{2ax(ax^2+d-ax^2-b)}{(ax^2+d)^2}$

$\dfrac{dy}{dx} = \dfrac{d-b}{(cx+d)^2}$, $\dfrac{dt}{dx} = \dfrac{2ax(d-b)}{(ax^2+d)^2}$

Using Chain Rule

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$

$$\frac{dy}{dt} = \frac{\cancel{d-b}}{(cx+d)^2} \times \frac{(ax^2+d)^2}{2ax\,\cancel{(d-b)}}$$

$$= \frac{(ax^2+d)^2}{2ax(cx+d)^2}$$

v) $\frac{x^2+1}{x^2-1}$ w.r.t. x^3

let

$y = \frac{x^2+1}{x^2-1}$, $t = x^3$

Differentiate w.r.t. 'x'

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$, $\frac{dt}{dx} = \frac{d}{dx}(x^3)$

$\frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$, $\frac{dt}{dx} = 3x^{3-1}\frac{d}{dx}(x)$

$\frac{dy}{dx} = \frac{(x^2-1)(2x+0) - (x^2+1)(2x-0)}{(x^2-1)^2}$, $\frac{dt}{dx} = 3x^2(1)$

$\frac{dy}{dx} = \frac{2x(\cancel{x^2}-1-\cancel{x^2}-1)}{(x^2-1)^2}$, $\frac{dt}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{2x(-2)}{(x^2-1)^2}$, $\frac{dt}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Using Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{-4x}{(x^2-1)^2} \times \frac{1}{3x^2}$$

$$= \frac{-4}{3x(x^2-1)^2}$$