Unit 4

Factorization and Algebraic Manipulation

EXERCISE 4.1

1. Factorize by identifying common factors.

   (i) $6x + 12$ (ii) $15y^2 + 20y$ (iii) $-12x^2 - 3x$

   (iv) $4a^2b + 8ab^2$ (v) $xy - 3x^2 + 2x$ (vi) $3a^2b - 9ab^2 + 15ab$

Solution:

(i) $6(x + 2)$ (ii) $5y(3y + 4)$ (iii) $-3x(4x + 1)$

(iv) $4ab(a + 2b)$ (v) $x(y - 3x + 2)$

(vi) $3ab(a - 3b + 5)$

2. Factorize and represent pictorially:

   (i) $5x + 15$ (ii) $x^2 + 4x + 3$ (iii) $x^2 + 6x + 8$

   (iv) $x^2 + 4x + 4$

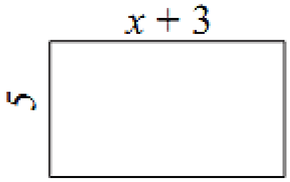
Solution

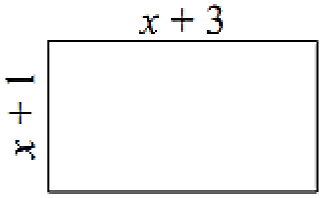
(i) $5(x+3)$

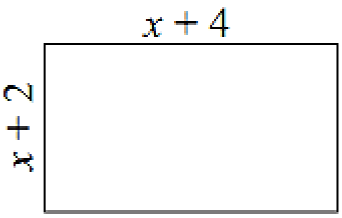
(ii) $x^2 + 4x + 3 = x^2 + 3x + x + 3 = x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$

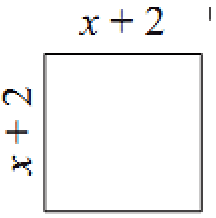
(iii) $x^2 + 6x + 8 = x^2 + 4x + 2x + 8 = x(x + 4) + 2(x + 4) = (x + 2)(x + 4)$

(iv) $x^2 + 4x + 4 = x^2 + 2x + 2x + 4 = x(x + 2) + 2(x + 2) = (x + 2)(x + 2) = (x + 2)^2$









3. Factorize:

(i) $x^2 + x - 12$ (ii) $x^2 + 7x + 10$ (iii) $x^2 - 6x + 8$

(iv) $x^2 - x - 56$ (v) $x^2 - 10x - 24$ (vi) $y^2 + 4y - 12$

(vii) $y^2 + 13y + 36$ (viii) $x^2 - x - 2$

Solution

(i) $x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x+4) - 3(x+4) = (x+4)(x-3)$

(ii) $x^2 + 7x + 10 = x^2 + 5x + 2x + 10 = x(x+5) + 2(x+5) = (x+5)(x+2)$

(iii) $x^2 - 6x + 8 = x^2 - 4x - 2x + 8 = x(x-4) - 2(x-4) = (x-4)(x-2)$

(iv) $x^2 - x - 56 = x^2 - 8x + 7x - 56 = x(x-8) + 7(x-8) = (x-8)(x+7)$

(v) $x^2 - 10x - 24 = x^2 - 12x + 2x - 24 = x(x-12) + 2(x-12) = (x-12)(x+2)$

(vi) $y^2 + 4y - 12 = y^2 + 6y - 2y - 12 = y(y+6) - 2(y+6) = (y+6)(y-2)$

(vii) $y^2 + 13y + 36 = y^2 + 9y + 4y + 36 = y(y+9) + 4(y+9) = (y+9)(y+4)$

(viii) $x^2 - x - 2 = x^2 - 2x + x - 2 = x(x-2) + 1(x-2) = (x-2)(x+1)$

4. Factorize:

(i) $2x^2 + 7x + 3$ (ii) $2x^2 + 11x + 15$ (iii) $4x^2 + 13x + 3$

(iv) $3x^2 + 5x + 2$ (v) $3y^2 - 11y + 6$ (vi) $2y^2 - 5y + 2$

(vii) $4z^2 - 11z + 6$ (viii) $6 + 7x - 3x^2$

Solution

(i) $2x^2 + 7x + 3 = 2x^2 + 6x + x + 3 = 2x(x+3) + 1(x+3) = (2x+1)(x+3)$

(ii) $2x^2 + 11x + 15 = 2x^2 + 6x + 5x + 15 = 2x(x+3) + 5(x+3) = (2x+5)(x+3)$

(iii) $4x^2 + 13x + 3 = 4x^2 + 12x + x + 3 = 4x(x+3) + 1(x+3) = (4x+1)(x+3)$

(iv) $3x^2 + 5x + 2 = 3x^2 + 3x + 2x + 2 = 3x(x+1) + 2(x+1) = (3x+2)(x+1)$

(v) $3y^2 - 11y + 6 = 3y^2 - 9y - 2y + 6 = 3y(y-3) - 2(y-3) = (3y-2)(y-3)$

(vi) $2y^2 - 5y + 2 = 2y^2 - 4y - y + 2 = 2y(y-2) - 1(y-2) = (2y-1)(y-2)$

(vii) $4z^2 - 11z + 6 = 4z^2 - 8z - 3z + 6 = 4z(z-2) - 3(z-2) = (4z-3)(z-2)$

(viii) $6 + 7x - 3x^2 = -3x^2 + 7x + 6 = -3x^2 + 9x - 2x + 6$

$= 3x(-x+3) + 2(-x+3) = (3x+2)(3-x)$

EXERCISE 4.2

1. Factorize each of the following expressions:

(i) $4x^4 + 81y^4$ (ii) $a^4 + 64b^4$ (iii) $x^4 + 4x^2 + 16$

(iv) $x^4 - 14x^2 + 1$ (v) $x^4 - 30x^2y^2 + 9y^4$ (vi) $x^4 - 11x^2y^2 + y^4$

Solution

1.(i) $4x^4 + 81y^4$

$= (2x^2)^2 + (9y^2)^2 = (2x^2)^2 + (9y^2)^2 + 2(2x^2)(9y^2) - 2(2x^2)(9y^2)$

$= (2x^2 + 9y^2)^2 - 36x^2y^2 = (2x^2 + 9y^2)^2 - (6xy)^2$

$= (2x^2 + 9y^2 - 6xy)(2x^2 + 9y^2 + 6xy)$

1.(ii) $a^4 + 64b^4$

$= (a^2)^2 + (8b^2)^2 = (a^2)^2 + (8b^2)^2 + 2(a^2)(8b^2) - 2(a^2)(8b^2)$

$= (a^2 + 8b^2)^2 - 16\,a^2b^2 = (a^2 + 8b^2)^2 - (4ab)^2$

$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$

1.(iii) $x^4 + 4x^2 + 16$

$= x^4 + 8x^2 - 4x^2 + 16 = x^4 + 8x^2 + 16 - 4x^2$

$= (x^2 + 4)^2 - (2x)^2$

$= (x^2 + 4 - 2x)(x^2 + 4 + 2x)$

1.(iv) $x^4 - 14x^2 + 1$

$= x^4 - 16x^2 + 2x^2 + 1 = x^4 + 2x^2 + 1 - 16x^2$

$= (x^2 + 1)^2 - (4x)^2$

$= (x^2 + 1 - 4x)(x^2 + 1 + 4x)$

1.(v) $\boldsymbol{x^4 - 30x^2y^2 \ + 9y^4}$

$= x^4 - 36x^2y^2 + 6x^2y^2 + 9y^4 = x^4 + 6x^2y^2 + 9y^4 - 36x^2y^2$

$= (x^2 + 3y^2)^2 \ - \ (6xy)^2$

$= (x^2 - 6xy + 3y^2)(x^2 + 6xy + 3y^2)$

1.(vi) $\boldsymbol{x^4 - 11x^2y^2 \ + y^4}$

$= x^4 - 9x^2y^2 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 9x^2y^2$

$= (x^2 + y^2)^2 \ - \ (3xy)^2$

$= (x^2 - 3xy + y^2)(x^2 + 3xy + y^2)$

2. Factorize each of the following expressions:

(i) $(x + 1)(x + 2)(x + 3)(x + 4) + 1$ (ii) $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

(iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$ (iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$ (vi) $(x + 1)(x - 1)(x + 2)(x - 2) + 13x^2$

Solution

2.(i) $\boldsymbol{(x + 1)(x + 2)(x + 3)(x + 4) + 1}$

$= (x + 1)(x + 4)(x + 2)(x + 3) + 1$

$= (x^2 + 5x + 4)(x^2 + 5x + 6) + 1$

$= (y + 4)(y + 6) + 1 = y^2 + 10y + 24 + 1 = y^2 + 10y + 25$

$= (y + 5)^2 = (x^2 + 5x + 5)^2$

2.(ii) $\boldsymbol{(x + 2)(x - 7)(x - 4)(x - 1) + 17}$

$= (x + 2)(x - 7)(x - 4)(x - 1) + 17$

$= (x^2 - 5x - 14)(x^2 - 5x + 4) + 17$

$= (y - 14)(y + 4) + 17 = y^2 - 10y - 56 + 17 = y^2 - 10y - 39$

$= y^2 - 13y + 3y - 39 = y(y - 13) + 3(y - 13)$

$= (y - 13)(y + 3) = (x^2 - 5x - 13)(x^2 - 5x + 3)$

2.(iii) $\boldsymbol{(2x^2+7x+3)(2x^2+7x+5)+1}$

$= (2x^2+7x+3)(2x^2+7x+5)+1$

$= (y+3)(y+5)+1 = y^2+8y+15+1 = y^2+8y+16$

$= (y+4)^2 = (2x^2+7x+4)^2$

2.(iv) $\boldsymbol{(3x^2+5x+3)(3x^2+5x+5)-3}$

$= (3x^2+5x+3)(3x^2+5x+5)-3$

$= (y+3)(y+5)-3 = y^2+8y+15-3 = y^2+8y+12$

$= y^2+6y+2y+12 = y(y+6)+2(y+6)$

$= (y+6)(y+2) = (3x^2+5x+6)(3x^2+5x+2)$

2.(v) $\boldsymbol{(x+1)(x+2)(x+3)(x+6)-3x^2}$

$= (x+1)(x+6)(x+2)(x+3)-3x^2$

$= (x^2+7x+6)(x^2+5x+6)-3x^2$

$= (x^2+6+7x)(x^2+6+5x)-3x^2$

$= (y+7x)(y+5x)-3x^2 = y^2+12xy+35x^2-3x^2$

$= y^2+12xy+32x^2 = y^2+8xy+4xy+32x^2 = y(y+8x)+4x(y+8x)$

$= (y+8x)(y+4x) = (x^2+8x+6)(x^2+4x+6)$

2.(vi) $\boldsymbol{(x+1)(x-1)(x+2)(x-2)+13x^2}$ **wrong statement**

$\boldsymbol{(x+1)(x-1)(x+2)(x-2)+5x^2}$ **right statement**

$= (x+1)(x+2)(x-1)(x-2)+5x^2$

$= (x^2+3x+2)(x^2-3x+2)+5x^2$

$= (x^2+2+3x)(x^2+2-3x)+5x^2$

$= (y+3x)(y-3x)+5x^2 = y^2-9x^2+5x^2 = y^2-4x^2$

$= (y-2x)(y+2x) = (x^2-2x+2)(x^2+2x+2)$

3. Factorize:

(i) $8x^3 + 12x^2 + 6x + 1$ (ii) $27a^3 + 108a^2b + 144ab^2 + 64b^3$

(iii) $x^3 + 48x^2y + 108xy^2 + 216y^3$ (iv) $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

Solution

Remember!

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

3.(i) $\boldsymbol{8x^3 + 12x^2 + 6x + 1}$

$= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$

$= (2x + 1)^3$

3.(ii) $\boldsymbol{27a^3 + 108a^2b + 144ab^2 + 64b^3}$

$= (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= (3a + 4b)^3$

3.(iii) $\boldsymbol{x^3 + 18x^2y + 108xy^2 + 216y^3}$ **wrong statement, i.e. use 18 instead 48**

$= (x)^3 + 3(x)^2(6y) + 3(x)(6y)^2 + (6y)^3$

$= (x + 6y)^3$

3.(iv) $\boldsymbol{8x^3 - 125y^3 + 150xy^2 - 60x^2y}$

$= (2x)^3 + (-5y)^3 + 3(2x)(-5y)^2 + 3(2x)^2(-5y)$

$= (2x - 5y)^3$

4. Factorize:

(i) $125a^3 - 1$ (ii) $64x^3 + 125$ (iii) $x^6 - 27$

(iv) $1000a^3 + 1$ (v) $343x^3 + 216$ (vi) $27 - 512y^3$

Solution

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

4.(i) $\boldsymbol{125a^3 - 1}$

$= (5a)^3 - (1)^3$

$= (5a - 1)[(5a)^2 + (5a)(1) + (1)^2]$

$= (5a - 1)(25a^2 + 5a + 1)$

4.(ii) $\mathbf{64x^3 + 125}$

$= (4x)^3 + (5)^3$

$= (4x + 5)[(4x)^2 - (4x)(5) + (5)^2]$

$= (4x + 5)(16x^2 - 20x + 25)$

4.(iii) $\mathbf{x^6 - 27}$

$= (x^2)^3 - (3)^3$

$= (x^2 - 3)[(x^2)^2 + (x^2)(3) + (3)^2]$

$= (x^2 - 3)(x^4 + 3x^2 + 9)$

4.(iv) $\mathbf{1000a^3 + 1}$

$= (10a)^3 + (1)^3$

$= (10a + 1)[(10a)^2 - (10a)(1) + (1)^2]$

$= (10a + 1)(100a^2 - 10a + 1)$

4.(v) $\mathbf{343x^3 + 216}$

$= (7x)^3 + (6)^3$

$= (7x + 6)[(7x)^2 - (7x)(6) + (6)^2]$

$= (7x + 6)(49x^2 - 42x + 36)$

4.(vi) $\mathbf{27 - 512y^3}$

$= (3)^3 - (8y)^3$

$= (3 - 8y)[(3)^2 + (3)(8y) + (8y)^2]$

$= (3 - 8y)(9 + 24y + 64y^2)$

EXERCISE 4.3

1. Find HCF by factorization method.

(i) $21x^2y, 35xy^2$ (ii) $4x^2 - 9y^2, 2x^2 - 3xy$

(iii) $x^3 - 1, x^2 + x + 1$ (iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$

(v) $t^2 + 3t - 4, t^2 + 5t + 4, t^2 - 1$ (vi) $x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x$

Solution

1.(i) $21x^2y, 35xy^2$

$21x^2y = 3 \times 7 \times x \times x \times y$

$35xy^2 = 5 \times 7 \times x \times y \times y$

$\text{HCF} = 7 \times x \times y = 7xy$

1.(ii) $4x^2 - 9y^2, 2x^2 - 3xy$

$4x^2 - 9y^2 = (2x)^2 - (3y)^2 = (2x - 3y)(2x + 3y)$

$2x^2 - 3xy = x(2x - 3y)$

$\text{HCF} = 2x - 3y$

1.(iii) $x^3 - 1, x^2 + x + 1$

$x^3 - 1 = (x - 1)(x^2 + x + 1)$

$x^2 + x + 1 = x^2 + x + 1$

$\text{HCF} = x^2 + x + 1$

1.(iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$

$a^3 + 2a^2 - 3a = a(a^2 + 2a - 3) = a(a^2 + 3a - a - 3) = a(a + 3)(a - 1)$

$2a^3 + 5a^2 - 3a = a(2a^2 + 5a - 3) = a(2a^2 + 6a - a - 3) = a(a + 3)(2a - 1)$

$\text{HCF} = a(a + 3)$

1.(v) $t^2-3t-4, t^2+5t+4, t^2-1$ **wrong statement in book**

$\mathrm{t^2-3t-4=t^2-4t+t-4=t(t-4)+1(t-4)=(t-4)(t+1)}$

$\mathrm{t^2+5t+4=t^2+4t+t+4=t(t+4)+1(t+4)=(t+4)(t+1)}$

$\mathrm{t^2-1=(t-1)(t+1)}$

$\mathrm{HCF=t+1}$

1.(vi) $x^2+15x+56, x^2+5x-24, x^2+8x$

$\mathrm{x^2+15x+56=x^2+8x+7x+56=x(x+8)+7(x+8)=(x+8)(x+7)}$

$\mathrm{x^2+5x-24=x^2+8x-3x-24=x(x+8)-3(x+8)=(x+8)(x-3)}$

$\mathrm{x^2+8x=x(x+8)}$

$\mathrm{HCF=x+8}$

2. Find HCF of the following expressions by using division method:

(i) $27x^3+9x^2-3x-9,\ 3x-2$ (ii) $x^3-9x^2+21x-15,\ x^2-4x+3$

(iii) $2x^3+2x^2+2x+2,\ 6x^3+12x^2+6x+12$

(iv) $2x^3-4x^2+6x,\ x^3-2x,\ 3x^2-6x$

Solution

2.(i) $27x^3+9x^2-3x-9, 3x-2$ **Replace with 9**

$$
\begin{array}{r|l|l}
 & 27x^3+9x^2-3x-10 & \\
3x-2 & \pm 27x^3 \mp 18x^2 & 9x^2+9x+5 \\
\hline
 & \quad 27x^2-3x-10 & \\
 & \quad \pm 27x^2 \mp 18x & \\
\hline
 & \qquad 15x-10 & \\
 & \qquad \pm 15x \mp 10 & \\
\hline
 & \qquad\quad 0 &
\end{array}
$$

$\mathrm{HCF=3x-2}$

2.(ii) $\boldsymbol{x^3-9x^2+21x-15, x^2-4x+3}$ **Replace by 15**

$$\begin{array}{l|c|l} & x^3-9x^2+21x-9\bullet & \\ x^2-4x+3 & \pm x^3 \mp 4x^2 \pm 3x & x-5 \\ \hline & -5x^2+18x-9 & \\ & \mp 5x^2 \pm 20x \mp 15 & \\ \hline & -2x+6 & \end{array}$$

we may write it $-2(x-3)$

Now

$$\begin{array}{l|c|l} & x^2-4x+3 & \\ x-3 & \pm x^2 \mp 3x & x-1 \\ \hline & -x+3 & \\ & \mp x \pm 3 & \\ \hline & 0 & \end{array}$$

$\mathrm{HCF} = \mathrm{x}-3$

2.(iii) $\boldsymbol{2x^3+2x^2+2x+2, 6x^3+12x^2+6x+12}$

or $\boldsymbol{2(x^3+x^2+x+1), 6(x^3+2x^2+x+2)}$

$$\begin{array}{l|c|l} & x^3+2x^2+x+2 & \\ x^3+x^2+x+1 & \pm x^3 \pm x^2 \pm x \pm 1 & 1 \\ \hline & x^2+1 & \end{array}$$

Now

$$\begin{array}{l|c|l} & x^3+x^2+x+1 & \\ x^2+1 & \pm x^3 \quad \pm x & x+1 \\ \hline & x^2+1 & \\ & \pm x^2 \pm 1 & \\ \hline & 0 & \end{array}$$

$\mathrm{HCF} = 2(x^2+1)$ **wrong answer in book**

2.(iv) $2x^3-4x^2+6x, x^3-2x, 3x^2-6x$

$$\begin{array}{l|l|l}
 & 2x^3-4x^2+6x & \\
x^3-2x & \pm 2x^3 \quad \mp 4x & 2 \\
\hline
 & -4x^2+10x & \text{we may write it } -2(2x^2-5x)
\end{array}$$

Now multiply x^3-2x with 2 and simplify

$$\begin{array}{l|l|l}
 & 2x^3-4x & \\
2x^2-5x & \pm 2x^3 \mp 5x^2 & x \\
\hline
 & 5x^2-4x &
\end{array}$$

Now multiply $5x^2-4x$ with 2 and simplify

$$\begin{array}{l|l|l}
 & 10x^2-8x & \\
2x^2-5x & \pm 10x^2 \mp 25x & 5 \\
\hline
 & 17x &
\end{array}$$

Now

$$\begin{array}{l|l|l}
 & 2x^2-5x & \\
x & \pm 2x^2 \mp 5x & 2x-5 \\
\hline
 & 0 &
\end{array}$$

And

$$\begin{array}{l|l|l}
 & 3x^2-6x & \\
x & \pm 3x^2 \mp 6x & 3x-6 \\
\hline
 & 0 &
\end{array}$$

HCF = x **wrong answer in book**

3. Find LCM of the following expressions by using prime factorization method.

(i) $2a^2b,\ 4ab^2,\ 6ab$ (ii) $x^2+x,\ x^3+x^2$

(iii) $a^2-4a+4,\ a^2-2a$ (iv) $x^4-16,\ x^3-4x$

(v) $16-4x^2,\ x^2+x-6,\ 4-x^2$

Solution

3.(i) $2a^2b, 4ab^2, 6ab$

$2a^2b = 2 \times a \times a \times b$

$4ab^2 = 2 \times 2 \times a \times b \times b$

$6ab = 2 \times 3 \times a \times b$

$\text{Common Factors} = 2 \times a \times b = 2ab$

$\text{non} - \text{ Common Factors} = 2 \times 3 \times a \times b = 6ab$

$\text{LCM} = \text{CF} \times \text{NCF} = 2ab \times 6ab = 12a^2b^2$

3.(ii) $x^2 + x, x^3 + x^2$

$x^2 + x = x(x+1)$

$x^3 + x^2 = x^2(x+1) = x \times x \times (x+1)$

$\text{Common Factors} = x(x+1)$

$\text{non} - \text{ Common Factors} = x$

$\text{LCM} = \text{CF} \times \text{NCF} = x(x+1) \times x = x^2(x+1)$

3.(iii) $a^2 - 4a + 4, a^2 - 2a$

$a^2 - 4a + 4 = (a-2)^2 = (a-2)(a-2)$

$a^2 - 2a = a(a-2)$

$\text{Common Factors} = (a-2)$

$\text{non} - \text{ Common Factors} = a(a-2)$

$\text{LCM} = \text{CF} \times \text{NCF} = (a-2) \times a(a-2) = a(a-2)^2$

3.(iv) $x^4 - 16, x^3 - 4x$

$\mathrm{x}^4 - 16 = (\mathrm{x}^2 - 4)(\mathrm{x}^2 + 4) = (\mathrm{x} - 2)(\mathrm{x} + 2)(\mathrm{x}^2 + 4)$

$\mathrm{x}^3 - 4\mathrm{x} = \mathrm{x}(\mathrm{x}^2 - 4) = \mathrm{x}(\mathrm{x} - 2)(\mathrm{x} + 2)$

$\text{Common Factors} = (\mathrm{x} - 2)(\mathrm{x} + 2)$

$\text{non} - \text{ Common Factors} = \mathrm{x}(\mathrm{x}^2 + 4)$

$\mathrm{LCM} = \mathrm{CF} \times \mathrm{NCF} = (\mathrm{x} - 2)(\mathrm{x} + 2) \times \mathrm{x}(\mathrm{x}^2 + 4) = \mathrm{x}(\mathrm{x}^4 - 16)$

3.(v) $16 - 4x^2, x^2 + x - 6, 4 - x^2$

$16 - 4\mathrm{x}^2 = 4(4 - \mathrm{x}^2) = 4(2 - \mathrm{x})(2 + \mathrm{x})$

$\mathrm{x}^2 + \mathrm{x} - 6 = \mathrm{x}^2 + 3\mathrm{x} - 2\mathrm{x} - 6 = (\mathrm{x} + 3)(\mathrm{x} - 2) = -(\mathrm{x} + 3)(2 - \mathrm{x})$

$4 - \mathrm{x}^2 = (2 - \mathrm{x})(2 + \mathrm{x})$

$\text{Common Factors} = (2 - \mathrm{x})(2 + \mathrm{x})$

$\text{non} - \text{ Common Factors} = -4(\mathrm{x} + 3)$

$\mathrm{LCM} = \mathrm{CF} \times \mathrm{NCF} = (2 - \mathrm{x})(2 + \mathrm{x}) \times -4(\mathrm{x} + 3) = 4(\mathrm{x}^2 - 4)(\mathrm{x} + 3)$

4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.

Solution

$\mathrm{HCF} = y - 7$

$\mathrm{LCM} = \mathrm{y}^3 - 10\mathrm{y}^2 + 11y + 70$

$\mathrm{p(y)} = \mathrm{y}^2 - 5y - 14$

$\mathrm{q(y)} = ?$

Using formula: $\mathrm{p(y)} \times \mathrm{q(y)} = \mathrm{HCF} \times \mathrm{LCM}$

$(\mathrm{y}^2 - 5y - 14) \times \mathrm{q(y)} = (y - 7) \times (\mathrm{y}^3 - 10\mathrm{y}^2 + 11y + 70)$

$\mathrm{q(y)} = \frac{(y-7)\times(\mathrm{y}^3-10\mathrm{y}^2+11y+70)}{(\mathrm{y}^2-5y-14)}$

$\mathrm{q(y)} = \frac{(y-7)\times(\mathrm{y}^3-10\mathrm{y}^2+11y+70)}{(y-7)(y+2)}$

$\mathrm{q(y)} = \frac{\mathrm{y}^3-10\mathrm{y}^2+11y+70}{y+2}$

$$\begin{array}{r|l|l}
 & y^3-10y^2+11y+70 & \\
y+2 & \pm y^3 \pm 2y^2 & y^2-12y+35 \\
\hline
 & \quad -12y^2+11y+70 & \\
 & \quad \mp 12y^2 \mp 24y & \\
\cline{2-2}
 & \qquad 35y+70 & \\
 & \qquad \pm 35y \pm 70 & \\
\cline{2-2}
 & \qquad\quad 0 &
\end{array}$$

$$\mathbf{q}(\mathbf{y})=\frac{\mathbf{y^3-10y^2+11}\boldsymbol{y}\mathbf{+70}}{\boldsymbol{y}\mathbf{+2}}=\mathbf{y^2-12}\boldsymbol{y}\mathbf{+35}$$

5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x+a)(x^3-a^3)$ and $x^2(x-a)$ respectively. If $p(x)=4x^2(x^2-a^2)$, find $q(x)$.

Solution

$\mathrm{HCF}=x^2(x-a)$

$\mathrm{LCM}=36x^3(x+a)(x^3-a^3)$

$p(x)=4x^2(x^2-a^2)$

$q(x)=?$

Using formula: $p(x)\times q(x)=\mathrm{HCF}\times\mathrm{LCM}$

$4x^2(x^2-a^2)\times q(x)=x^2(x-a)\times 36x^3(x+a)(x^3-a^3)$

$q(x)=\frac{x^2(x-a)\times 36x^3(x+a)(x^3-a^3)}{4x^2(x^2-a^2)}$

$q(x)=9x^3(x^3-a^3)$

6. The HCF and LCM of two polynomials is $(x+a)$ and $12x^2(x+a)(x^2-a^2)$ respectively. Find the product of the two polynomials.

Solution

$\mathrm{HCF}=(x+a)$

$\mathrm{LCM}=12x^2(x+a)(x^2-a^2)$

$p(x)\times q(x)=?$

Using formula: $p(x)\times q(x)=\mathrm{HCF}\times\mathrm{LCM}$

$p(x)\times q(x)=(x+a)\times 12x^2(x+a)(x^2-a^2)$

$p(x)\times q(x)=12x^2(x+a)^3(x-a)$ **wrong answer in book**

EXERCISE 4.4

1. Find the square root of the following polynomials by factorization method:

(i) $x^2 - 8x + 16$ (ii) $9x^2 + 12x + 4$

(iii) $36a^2 + 84a + 49$ (iv) $64y^2 - 32y + 4$

(v) $200t^2 - 120t + 18$ (vi) $40x^2 + 120x + 90$

Solution

1.(i) $\sqrt{\boldsymbol{x^2 - 8x + 16}}$ =???

$$x^2 - 8x + 16 = (x)^2 - 2(x)(4) + (4)^2 = (x-4)^2$$

$$\sqrt{x^2 - 8x + 16} = \sqrt{(x-4)^2}$$

$$\sqrt{x^2 - 8x + 16} = \pm(x-4)$$

1.(ii) $\sqrt{\boldsymbol{9x^2 + 12x + 4}}$ =???

$$9x^2 + 12x + 4 = (3x)^2 + 2(3x)(2) + (2)^2 = (3x+2)^2$$

$$\sqrt{9x^2 + 12x + 4} = \sqrt{(3x+2)^2}$$

$$\sqrt{9x^2 + 12x + 4} = \pm(3x+2)$$

1.(iii) $\sqrt{\boldsymbol{36a^2 + 84a + 49}}$ =???

$$36a^2 + 84a + 49 = (6a)^2 + 2(6a)(7) + (7)^2 = (6a+7)^2$$

$$\sqrt{36a^2 + 84a + 49} = \sqrt{(6a+7)^2}$$

$$\sqrt{36a^2 + 84a + 49} = \pm(6a+7)$$

1.(iv) $\sqrt{\boldsymbol{64y^2 - 32y + 4}}$ =???

$$64y^2 - 32y + 4 = (8y)^2 - 2(8y)(2) + (2)^2 = (8y-2)^2$$

$$\sqrt{64y^2 - 32y + 4} = \sqrt{(8y-2)^2}$$

$$\sqrt{64y^2 - 32y + 4} = \pm(8y-2)$$

1.(v) $\sqrt{\mathbf{200t^2 - 120t + 18}}$ =???

$200\mathrm{t}^2 - 120\mathrm{t} + 18 = 2[100\mathrm{t}^2 - 60\mathrm{t} + 9]$

$200\mathrm{t}^2 - 120\mathrm{t} + 18 = 2[(10\mathrm{t})^2 - 2(10\mathrm{t})(3) + (3)^2] = 2(10\mathrm{t} - 3)^2$

$\sqrt{64\mathrm{y}^2 - 32\mathrm{y} + 18} = \sqrt{2(10\mathrm{t} - 3)^2}$

$\sqrt{64\mathrm{y}^2 - 32\mathrm{y} + 18} = \pm\sqrt{2}(10\mathrm{t} - 3)$

1.(vi) $\sqrt{\mathbf{40x^2 + 120x + 90}}$ =???

$40\mathrm{x}^2 + 120\mathrm{x} + 90 = 10(4\mathrm{x}^2 + 12\mathrm{x} + 9) = 10[(2\mathrm{x})^2 + 2(2\mathrm{x})(3) + (3)^2]$

$40\mathrm{x}^2 + 120\mathrm{x} + 90 = 10(2\mathrm{x} + 3)^2$

$\sqrt{40\mathrm{x}^2 + 120\mathrm{x} + 90} = \sqrt{10(2\mathrm{x} + 3)^2}$

$\sqrt{40\mathrm{x}^2 + 120\mathrm{x} + 90} = \pm\sqrt{10}(2\mathrm{x} + 3)$

2. Find the square root of the following polynomials by division method:
   - (i) $4x^4 - 28x^3 + 37x^2 + 42x + 9$
   - (ii) $121x^4 - 198x^3 - 183x^2 + 216x + 144$
   - (iii) $x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$
   - (iv) $4x^4 - 12x^3 + 37x^2 - 42x + 49$

Solution

1.(i) $\sqrt{\mathbf{4x^4 - 28x^3 + 37x^2 + 42x + 9}}$ =???

$$\begin{array}{r|l}
 & 2\mathrm{x}^2 - 7\mathrm{x} - 3 \\
\hline
 & 4\mathrm{x}^4 - 28\mathrm{x}^3 + 37\mathrm{x}^2 + 42\mathrm{x} + 9 \\
2\mathrm{x}^2 & \pm 4\mathrm{x}^4 \\
\hline
 & -28\mathrm{x}^3 + 37\mathrm{x}^2 \\
4\mathrm{x}^2 - 7\mathrm{x} & \mp 28\mathrm{x}^3 \pm 49\mathrm{x}^2 \\
\hline
 & -12\mathrm{x}^2 + 42\mathrm{x} + 9 \\
4\mathrm{x}^2 - 14\mathrm{x} - 3 & \mp 12\mathrm{x}^2 \pm 42\mathrm{x} \pm 9 \\
\hline
 & 0
\end{array}$$

$\sqrt{4\mathrm{x}^4 - 28\mathrm{x}^3 + 37\mathrm{x}^2 + 42\mathrm{x} + 9} = \pm(2\mathrm{x}^2 - 7\mathrm{x} - 3)$

1.(ii) $\sqrt{\mathbf{121x^4 - 198x^3 - 183x^2 + 216x + 144}}$ =???

	$11x^2 - 9x - 12$
$11x^2$	$121x^4 - 198x^3 - 183x^2 + 216x + 144$ $\pm 121x^4$
$22x^2 - 9x$	$-198x^3 - 183x^2$ $\mp 198x^3 \pm 81x^2$
$22x^2 - 18x - 12$	$-264x^2 + 216x + 144$ $\mp 264x^2 \pm 216x \pm 144$
	0

$$\sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144} = \pm(11x^2 - 9x - 12)$$

1.(iii) $\sqrt{\mathbf{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4}}$ =???

	$x^2 - 5xy + y^2$
x^2	$x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$ $\pm x^4$
$2x^2 - 5xy$	$-10x^3y + 27x^2y^2$ $\mp 10x^3y \pm 25x^2y^2$
$2x^2 - 10xy + y^2$	$2x^2y^2 - 10xy^3 + y^4$ $\pm 2x^2y^2 \mp 10xy^3 \pm y^4$
	0

$$\sqrt{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4} = \pm(x^2 - 5xy + y^2)$$

1.(iv) $\sqrt{\mathbf{4x^4 - 12x^3 + 37x^2 - 42x + 49}}$ =???

	$2x^2 - 3x + 7$
$2x^2$	$4x^4 - 12x^3 + 37x^2 - 42x + 49$ $\pm 4x^4$
$4x^2 - 3x$	$-12x^3 + 37x^2$ $\mp 12x^3 \pm 9x^2$
$4x^2 - 6x + 7$	$28x^2 - 42x + 49$ $\pm 28x^2 \mp 42x \pm 49$
	0

$$\sqrt{4x^4 - 12x^3 + 37x^2 - 42x + 49} = \pm(2x^2 - 3x + 7)$$

3. An investor's return $R(x)$ in rupees after investing x thousand rupees is given by quadratic expression:

$$R(x) = -x^2 + 6x - 8$$

Factorize the expression and find the investment levels that result in zero return.

Solution

$\mathrm{R(x)} = -\mathrm{x}^2 + 6\mathrm{x} - 8 = -\mathrm{x}^2 + 4\mathrm{x} + 2\mathrm{x} - 8 = -\mathrm{x}(\mathrm{x} - 4) + 2(\mathrm{x} - 4)$

$\mathrm{R(x)} = (-\mathrm{x} + 2)(\mathrm{x} - 4)$

For zero return $\mathrm{R(x)} = 0$ we have $(-\mathrm{x} + 2)(\mathrm{x} - 4) = 0$

$-\mathrm{x} + 2 = 0$ or $\mathrm{x} - 4 = 0$

$\mathrm{x} = 2$ or $\mathrm{x} = 4$

Investment levels that result in zero return will be $\mathrm{x} = 2$ and $\mathrm{x} = 4$

4. A company's profit $P(x)$ in rupees from selling x units of a product is modeled by the cubic expression:

$$P(x) = x^3 - 15x^2 + 75\,x - 125$$

Find the break-even point(s), where the profit is zero.

Solution

$\mathrm{P(x)} = \mathrm{x}^3 - 15\mathrm{x}^2 + 75x - 125$

$\mathrm{P(x)} = (\mathrm{x})^3 - 3(\mathrm{x})^2(5) + 3(\mathrm{x})(5)^2 - (5)^3 = (\mathrm{x} - 5)^3$

Since profit is zero, using $\mathrm{P(x)} = 0$ we have $(\mathrm{x} - 5)^3 = 0$

After taking cube root on both sides we have $\mathrm{x} = 5$

5. The potential energy $V(x)$ in an electric field varies as a cubic function of distance x, given by:

$$V(x) = 2x^3 - 6x^2 + 4x$$

Determine where the potential energy is zero.

Solution

$\mathrm{V(x)} = 2\mathrm{x}^3 - 6\mathrm{x}^2 + 4x$

$\mathrm{V(x)} = 2\mathrm{x}(\mathrm{x}^2 - 3\mathrm{x} + 2) = 2\mathrm{x}(\mathrm{x} - 2)(\mathrm{x} - 1)$

For zero potential energy, using $\mathrm{V(x)} = 0$ we have $2\mathrm{x}(\mathrm{x} - 1)(\mathrm{x} - 2) = 0$

Then $\mathrm{x} = 0, \mathrm{x} = 1, \mathrm{x} = 2$

6. In structural engineering, the deflection $Y(x)$ of a beam is given by:

$$Y(x) = 2x^2 - 8x + 6$$

This equation gives the vertical deflection at any point x along the beam. Find the points of zero deflection.

Solution

$Y(x) = 2x^2 - 8x^2 + 6$

$Y(x) = 2(x^2 - 4x + 3)$

$Y(x) = 2(x^2 - 3x - x + 3)$

$Y(x) = 2(x - 1)(x - 3)$

For zero potential deflection, using $Y(x) = 0$ we have $2(x - 1)(x - 3) = 0$

$2 \neq 0$ then $x - 1 = 0$ or $x - 3 = 0$

Then $x = 1, x = 3$

REVIEW EXERCISE 4

1. Four options are given against each statement. Encircle the correct option.

   i. The factorization of $12x + 36$ is:

   [✓ (a) $12(x + 3)$] (b) $12(3x)$ (c) $12(3x + 1)$ (d) $x(12 + 36x)$

   ii. The factors of $4x^2 - 12y + 9$ are:

   (a) $(2x + 3)^2$ [✓ (b) $(2x - 3)^2$]

   (c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$

   iii. The HCF of a^3b^3 and ab^2 is:

   (a) a^3b^3 [✓ (b) ab^2] (c) a^4b^5 (d) a^2b

iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

(a) $480x^3y$ (b) $240xy$ [✓ (c) $240x^2y$] (d) $120x^4y$

v. Product of LCM and HCF = __________ of two polynomials.

(a) sum (b) difference [✓ (c) product] (d) quotient

vi. The square root of $x^2 - 6x + 9$ is:

[✓ (a) $\pm(x - 3)$] (b) $\pm(x + 3)$ (c) $x - 3$ (d) $x + 3$

vii. The LCM of $(a - b)^2$ and $(a - b)^4$ is:

(a) $(a - b)^2$ (b) $(a - b)^3$ [✓ (c) $(a - b)^4$] (d) $(a - b)^6$

viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

[✓ (a) $(x + 1)^3$] (b) $(x - 1)^3$

(c) $(x + 1)(x^2 + x + 1)$ (d) $(x - 1)(x^2 - x + 1)$

ix. Cubic polynomial has degree:

(a) 1 (b) 2 [✓ (c) 3] (d) 4

x. One of the factors of $x^3 - 27$ is:

[✓ (a) $x - 3$] (b) $x + 3$ (c) $x^2 - 3x + 9$ (d) Both *a* and *c*

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$ (viii) $(x + 3)(x + 4)(x + 5)(x + 6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

Solution

2.(i) $\mathbf{4x^3 + 18x^2 - 12x}$

$= 2x(2x^2 + 9x - 6)$

2.(ii) $\mathbf{x^3 + 64y^3}$

$= (x)^3 + (4y)^3$

$= (x + 4y)[(x)^2 - (x)(4y) + (4y)^2]$

$= (x + 4y)(x^2 - 4xy + 16y^2)$

2.(iii) $\mathbf{x^3y^3 - 8}$

$= (xy)^3 - (2)^3$

$= (xy - 2)[(xy)^2 + (xy)(2) + (2)^2]$

$= (xy - 2)(x^2y^2 + 2xy + 4)$

2.(iv) $\mathbf{-x^2 - 23x - 60}$

$= -(x^2 + 23x + 60) = -[x^2 + 20x + 3x + 60] = -[x(x + 20) + 3(x + 20)]$

$= -(x + 3)(x + 20)$

2.(v) $\mathbf{2x^2 + 7x + 3}$

$= 2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3)$

$= (2x + 1)(x + 3)$

2.(vi) $\boldsymbol{x^4 + 64}$

$= (x^2)^2 + (8)^2 + 2(x^2)(8) - 2(x^2)(8) = (x^2 + 8)^2 - 16x^2$

$= (x^2 + 8)^2 - (4x)^2 = (x^2 + 8 - 4x)(x^2 + 8 + 4x)$

$= (x^2 - 4x + 8)(x^2 + 4x + 8)$

2.(vii) $\boldsymbol{x^4 + 2x^2 + 9}$

$= (x^2)^2 + 2x^2 + (3)^2$

$= (x^2)^2 + 2(x^2)(3) + (3)^2 + 2x^2 - 2(x^2)(3)$

$= (x^2 + 3)^2 - 4x^2$

$= (x^2 + 3)^2 - (2x)^2 = (x^2 + 3 - 2x)(x^2 + 3 + 2x)$

$= (x^2 - 2x + 3)(x^2 + 2x + 3)$

2.(viii) $\boldsymbol{(x + 3)(x + 4)(x + 5)(x + 6) - 360}$

$= (x + 3)(x + 6)(x + 4)(x + 5) - 360$

$= (x^2 + 9x + 18)(x^2 + 9x + 20) - 360$

$= (y + 18)(y + 20) - 360 = y^2 + 38y + 360 - 360 = y^2 + 38y$

$= y(y + 38) = (x^2 + 9x)(x^2 + 9x + 38)$

$= x(x + 9)(x^2 + 9x + 38)$

2.(ix) $\boldsymbol{(x^2 + 6x + 3)(x^2 + 6x - 9) + 36}$

$= (x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

$= (y + 3)(y - 9) + 36 = y^2 - 6y - 27 + 36 = y^2 - 6y + 9$

$= (y)^2 - 2(y)(3) + (3)^2 = (y - 3)^2$

$= (x^2 + 6x - 3)^2$

3. Find LCM and HCF by prime factorization method:

(i) $4x^3 + 12x^2,\ 8x^2 + 16x$ (ii) $x^3 + 3x^2 - 4x,\ x^2 - x - 6$

(iii) $x^2 + 8x + 16, x^2 - 16$ (iv) $x^3 - 9x,\ x^2 - 4x + 3$

Solution

3.(i) $4x^3 + 12x^2, 8x^2 + 16x$

$4x^3 + 12x^2 = 4x^2(x + 3) = 4 \times x \times x \times (x + 3)$

$8x^2 + 16x = 8x(x + 2) = 2 \times 4 \times x \times (x + 2)$

$\text{Common Factors} = 4x$

$\text{Un} - \text{ Common Factors} = x \times (x + 3) \times 2 \times (x + 2) = 2x(x + 3)(x + 2)$

$\text{HCF} = 4x$

$\text{LCM} = \text{CF} \times \text{UCF} = 4x \times 2x(x + 3)(x + 2) = 8x^2(x + 2)(x + 3)$

3.(ii) $x^3 + 3x^2 - 4x, x^2 - x - 6$

$x^3 + 3x^2 - 4x = x(x^2 + 3x - 4) = x(x^2 + 4x - x - 4) = x(x - 1)(x + 4)$

$x^2 - x - 6 = x^2 - 3x + 2x - 6 = (x - 3)(x + 2)$

$\text{Common Factors} = 1$

$\text{Un} - \text{ Common Factors} = x(x - 1)(x + 4)(x - 3)(x + 2)$

$\text{HCF} = 1$

$\text{LCM} = \text{CF} \times \text{UCF} = x(x - 1)(x + 2)(x - 3)(x + 4)$ **wrong answer in book**

3.(iii) $x^2 + 8x + 16, x^2 - 16$

$x^2 + 8x + 16 = (x)^2 + 2(x)(4) + (4)^2 = (x + 4)^2 = (x + 4)(x + 4)$

$x^2 - 16 = (x)^2 - (4)^2 = (x - 4)(x + 4)$

$\text{Common Factors} = (x + 4)$

$\text{Un} - \text{ Common Factors} = (x - 4)(x + 4)$

$\text{HCF} = (x + 4)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x + 4) \times (x - 4)(x + 4) = (x - 4)(x + 4)^2$

3.(iv) $\boldsymbol{x^3 - 9x, x^2 - 4x + 3}$

$x^3 - 9x = x(x^2 - 9) = x(x - 3)(x + 3)$

$x^2 - 4x + 3 = x^2 - 3x - x + 3 = (x - 3)(x - 1)$

$\text{Common Factors} = (x - 3)$

$\text{Un} - \text{ Common Factors} = x(x + 3)(x - 1)$

$\text{HCF} = (x - 3)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x - 3) \times x(x + 3)(x - 1)$

$\text{LCM} = x(x - 1)(x^2 - 9)$ **wrong answer in book**

4. Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.

Solution

Factorization:

$16x^4 + 8x^2 + 1 = (4x^2)^2 + 2(4x^2)(1) + (1)^2$

$16x^4 + 8x^2 + 1 = (4x^2 + 1)^2$

$\sqrt{16x^4 + 8x^2 + 1} = \sqrt{(4x^2 + 1)^2}$

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2 + 1)$

Division Method:

$$\begin{array}{c|l} & 4x^2 + 1 \\ \hline & 16x^4 + 8x^2 + 1 \\ 4x^2 & 16x^4 \\ \hline & \quad 8x^2 + 1 \\ 8x^2 + 1 & \quad \pm 8x^2 \pm 1 \\ \hline & \qquad 0 \end{array}$$

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2 + 1)$

5. Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where x represents the number of years. What is the optimal repayment period for Huria's loan?

Solution

$C(x) = x^2 - 8x + 15 = x^2 - 5x - 3x + 15 = (x - 5)(x - 3)$

For optimal repayment, using $C(x) = 0$ we have $(x - 5)(x - 3) = 0$

We have $x = 3, x = 5$

. That is 3 years or 5 years.