Al Huda Science Academy Notes

10th Mathematics — New Book
Chapter 1: Complex Numbers

Exercise 1.4 Full Solution

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Board Exam Preparation Series

Q.1 — Find the real and imaginary parts of the following complex numbers:

(i) $(8 - 3i)^2$

$= (8 - 3i)(8 - 3i) = 64 - 24i - 24i + 9i^2 = 64 - 48i + 9(-1)$

$= 64 - 48i - 9 = 55 - 48i$

Real Part = 55, Imaginary Part = −48

(ii) $(5 + 3i)^{-1}$

$= 1/(5+3i) \times (5-3i)/(5-3i) = (5-3i)/[(5)^2-(3i)^2] = (5-3i)/(25+9)$

Real Part = 5/34, Imaginary Part = −3/34

(iii) $(4 - 5i)^{-1}$

$= 1/(4-5i) \times (4+5i)/(4+5i) = (4+5i)/[(4)^2+(5)^2] = (4+5i)/(16+25)$

Real Part = 4/41, Imaginary Part = 5/41

(iv) $(4 - 3i)^{-2}$

$= (1/(4-3i))^2 = ((4+3i)/[(4)^2+(3)^2])^2 = ((4+3i)/25)^2 = (4+3i)^2/625$

$= (16 + 24i + 9i^2)/625 = (16 + 24i - 9)/625 = (7 + 24i)/625$

Real Part = 7/625, Imaginary Part = 24/625

(v) $((3 + 2i)/(4 + 3i))^{-1}$

$= (4+3i)/(3+2i) = (4+3i)/(3+2i) \times (3-2i)/(3-2i) = [(4+3i)(3-2i)] / [(3)^2+(2)^2]$

$= (12 - 8i + 9i - 6i^2)/13 = (12 + i + 6)/13 = (18+i)/13$

Real Part = 18/13, Imaginary Part = 1/13

(vi) $((2 - i)/(2 + i))^{-2}$

$= ((2+i)/(2-i))^2 = [(2+i)^2/((2)^2+(1)^2)]^2 = [(4 + 4i + i^2)/5]^2 = [(3+4i)/5]^2$

$= (3+4i)^2/25 = (9 + 24i + 16i^2)/25 = (-7 + 24i)/25$

Real Part = −7/25, Imaginary Part = 24/25

(vii) $((1 - 2i)/(1 + i))^2$

First simplify the fraction:

$(1-2i)/(1+i) = [(1-2i)(1-i)] / [(1+i)(1-i)] = (1 - i - 2i + 2i^2)/(1 - i^2)$

$= (1 - 3i - 2)/2 = (-1 - 3i)/2$

Now square the result:

$((-1-3i)/2)^2 = (-1-3i)^2/2^2 = (1 + 6i + 9i^2)/4$

$= (1 + 6i - 9)/4 = (-8 + 6i)/4 = -2 + (3/2)i$

Real Part = −2, Imaginary Part = 3/2

Q.2 — Solve the following simultaneous linear equations with complex coefficients for w and z:

(i) $3z + (2 + i)w = 11 - i$ and $(2 - i)z - w = -1 + i$

From the 2nd equation:

$(2-i)z - w = -1+i \Rightarrow w = (2-i)z + 1 - i$

Substitute into the 1st equation:

$3z + (2+i)[(2-i)z + 1-i] = 11-i$

$3z + (2+i)(2-i)z + (2+i)(1-i) = 11-i$

$3z + 5z + (3-i) = 11-i$ [since $(2+i)(2-i)=5$]

$8z + 3 - i = 11 - i \Rightarrow 8z = 8 \Rightarrow z = 1$

Find w using $w = (2-i)z + 1 - i$:

$w = (2-i)(1) + 1 - i = 2 - i + 1 - i = 3 - 2i$

$z = 1$, $w = 3 - 2i$

(ii) $2z + (3 + i)w = 9 - i$ and $-iz - iw = -1 + i$

From the 2nd equation:

$-iz - iw = -1+i \Rightarrow -i(z+w) = -1+i \Rightarrow z+w = (-1+i)/(-i) = -1-i$

$\therefore z = -1 - i - w$

Substitute into the 1st equation:

$2(-1-i-w) + (3+i)w = 9-i \Rightarrow -2-2i-2w + 3w + iw = 9-i$

$-2-2i + w + iw = 9-i \Rightarrow w(1+i) = 9-i+2+2i = 11+i$

Solve for w by multiplying by the conjugate $(1-i)$:

$w = (11+i)/(1+i) \times (1-i)/(1-i) = [(11+i)(1-i)] / [1^2-i^2]$

$= (11 - 11i + i - i^2)/2 = (12 - 10i)/2 = 6 - 5i$

Find $z = -1 - i - w$:

$z = -1 - i - (6-5i) = -1-i-6+5i = -7 + 4i$

$z = -7 + 4i$, $w = 6 - 5i$

(iii) $z - 4w = 3i$ and $2z + 3w = 11 - 5i$

From the 1st equation:

$z = 4w + 3i$

Substitute into the 2nd equation:

$2(4w+3i) + 3w = 11-5i \Rightarrow 8w + 6i + 3w = 11-5i$

$11w = 11 - 5i - 6i = 11 - 11i \Rightarrow w = (11-11i)/11 = 1 - i$

Find $z = 4w + 3i$:

$z = 4(1-i) + 3i = 4 - 4i + 3i = 4 - i$

$z = 4 - i$, $w = 1 - i$

(iv) $z + w = 3i$ and $2z + 3w = 2$

From the 1st equation:

$z = 3i - w$

Substitute into the 2nd equation:

$2(3i - w) + 3w = 2 \Rightarrow 6i - 2w + 3w = 2 \Rightarrow w = 2 - 6i$

Find $z = 3i - w$:

$z = 3i - (2 - 6i) = 3i - 2 + 6i = -2 + 9i$

$z = -2 + 9i, \quad w = 2 - 6i$

(v) $2z + (3 - i)w = 1$ and $z - (1 - i)w = 2$

From the 2nd equation:

$z = 2 + (1 - i)w$

Substitute into the 1st equation:

$2[2 + (1 - i)w] + (3 - i)w = 1$

$4 + 2(1 - i)w + (3 - i)w = 1$

$4 + [(2 - 2i) + (3 - i)]w = 1 \Rightarrow 4 + (1 + 3i)w = 5$

So $(1+3i)w = 1 + 4 = 5$. Solve for w by multiplying by the conjugate $(1-3i)$:

$w = 5/(1+3i) \times (1-3i)/(1-3i) = 5(1-3i)/[1^2+3^2] = 5(1-3i)/10 = 1/2 - (3/2)i$

Find $z = 2 - (1-i)w$:

$z = 2 - (1-i)(1/2 - (3/2)i)$

$= 2 - [1/2 - (3/2)i - (1/2)i + (3/2)i^2]$

$= 2 - [1/2 - 2i - 3/2] = 2 - [-1 - 2i] = 2 + 1 + 2i = -1 + 2i$

$z = -1 + 2i, \quad w = 1/2 - (3/2)i$

KEY FACTS — Exercise 1.4

Powers of i

$i^0=1$ $i^1=i$ $i^2=-1$ $i^3=-i$ $i^4=1$

Conjugate

If $z = a + bi$, then $\bar{z} = a - bi$

Important identities

$(a+bi)(a-bi) = a^2 + b^2$

$(a+bi)^2 = (a^2-b^2) + 2abi$

$(a-bi)^2 = (a^2-b^2) - 2abi$

Inverse of a complex number

If $z = a+bi$ $(a^2+b^2 \neq 0)$, then $1/z = (a-bi)/(a^2+b^2)$

Real & imaginary part

For $z = a + bi$: Real Part $= a$, Imaginary Part $= b$

General rules

$z_1 + z_2 = (a+bi) + (c+di) = (a+c) + (b+d)i$

$z_1 - z_2 = (a+bi) - (c+di) = (a-c) + (b-d)i$

$z_1z_2 = (a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$z_1/z_2 = (z_1\bar{z}_2)/(z_2\bar{z}_2)$